MATLAB Function to Design Second Order Filters

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1 Introduction

Second order filters, sometimes called *second order sections* (SOS), are useful building blocks for audio signal processing. Three common SOS types are

- Bass shelf filter - boosts or cuts frequency response around and below a transition frequency.
- Treble shelf filter - boosts or cuts frequency response around and above a transition frequency.
- Peak filter - boosts or cuts frequency response around a peak frequency.

At other frequencies, e.g., above the transition frequency for a bass shelf filter, an SOS is transparent, i.e., the amplitude of the frequency response is 0 dB. Thus, a cascade of SOS can boost or cut frequency response at several frequencies. This property makes a SOS an attractive component for an audio equalizer.

The MATLAB function `order2filt` uses a straightforward algorithm to determine the coefficients of a SOS based on the filter type, sampling frequency, dB gain, peak or transition frequency, and a Q factor.

2 Filter Design Algorithm

We can use a two-step process to design a digital filter:

1. Obtain the transfer function for the corresponding analog filter $H(s)$.
2. Apply the bilinear transform to determine the digital filter transfer function $H(z)$.

The MATLAB function `bilinear`, which is included in the MATLAB signal processing toolbox, accomplishes the second step, so we need only determine $H(s)$.

2.1 Shelf Filters

Consider a biquadratic transfer function with poles and zeros given by

$$\begin{aligned} p &= -\sqrt{b}e^{\pm j\pi/4} \\ z &= -\sqrt{a}e^{\pm j\pi/4} \end{aligned} \tag{1}$$

The transfer function is given by

$$\begin{aligned} H(s) &= K\frac{(s+\sqrt{a}e^{j\pi/4})(s+\sqrt{a}e^{-j\pi/4})}{(s+\sqrt{b}e^{j\pi/4})(s+\sqrt{b}e^{-j\pi/4})} \\ &= K\frac{s^2+\sqrt{2a}s+a}{s^2+\sqrt{2b}s+b} \end{aligned} \tag{2}$$

where K is an arbitrary constant. If we choose $K = b$, then the magnitude of $H(s)$ is a at low frequencies and b at high frequecies. For $a = 1$ and $b = g$, $H(s)$ is a treble shelf filter with gain g. For $a = g$ and $b = 1$, $H(s)$ is a bass shelf filter. Figure 1 shows the frequency response for $a = 1$ and $b = g$ (treble shelf filter) for several values of g (derived from given dB values). Note the lack of symmetry in the frequency response curves, e.g., the curve for -5 dB is not the mirror image of the curve for $+5$ dB. We can fix this by inverting $H(s)$ and replacing g with $1/g$ for $g < 1$ (< 0 dB). This is equivalent to calculating the gain factor g as

$$g = 10^{|G|/20} \tag{3}$$

and inverting $H(s)$ for $G < 0$. The result is shown in Figure 2. Figure 3 shows a similar result for a bass shelf filter.

The poles and zeros of our shelf filters lie along a line at an angle of $\pm\pi/4$ relative to the negative real axis. If we change this angle to a general angle θ, we have

$$\begin{aligned} p &= -\sqrt{b}e^{\pm j\theta} \\ z &= -\sqrt{a}e^{\pm j\theta} \end{aligned} \tag{4}$$

and our transfer function becomes

$$H(s) == b\frac{s^2+2\sqrt{a}\cos\theta s+a}{s^2+2\sqrt{b}\cos\theta s+b} \tag{5}$$

Figure 4 shows the frequency response of a bass boost filter with gain $G = 10$ dB and various values of θ. As θ approaches $\pi/2$, we get peaking, which is probably not desirable, and for $\theta \geq \pi/2$, the filter becomes unstable. However, for θ up to about $\pi/3$, we get an increasing slope with no significant peaking.

Since the value of $\cos\theta$ varies from 1 to 0 as θ varies from 0 to $\pi/2$, we can also express $H(s)$ as

$$H(s) == b\frac{s^2+2\sqrt{a}s/Q+a}{s^2+2\sqrt{b}s/Q+b} \tag{6}$$

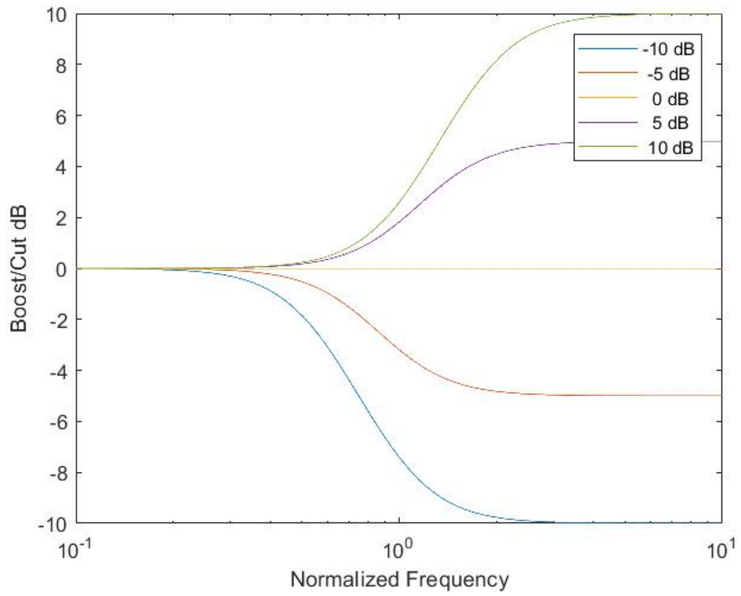

Figure 1: Frequency Response of Treble Shelf Filter

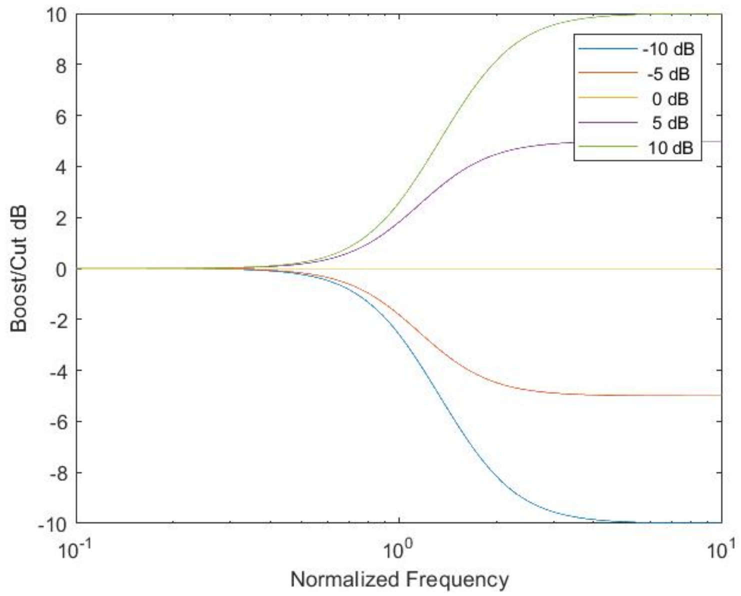

Figure 2: Frequency Response of Symmetric Treble Shelf Filter

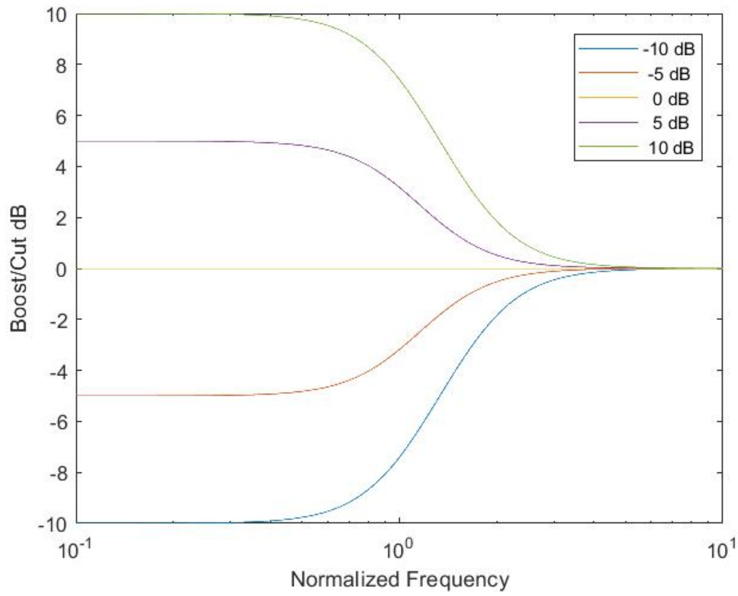

Figure 3: Frequency Response of Symmetric Bass Shelf Filter

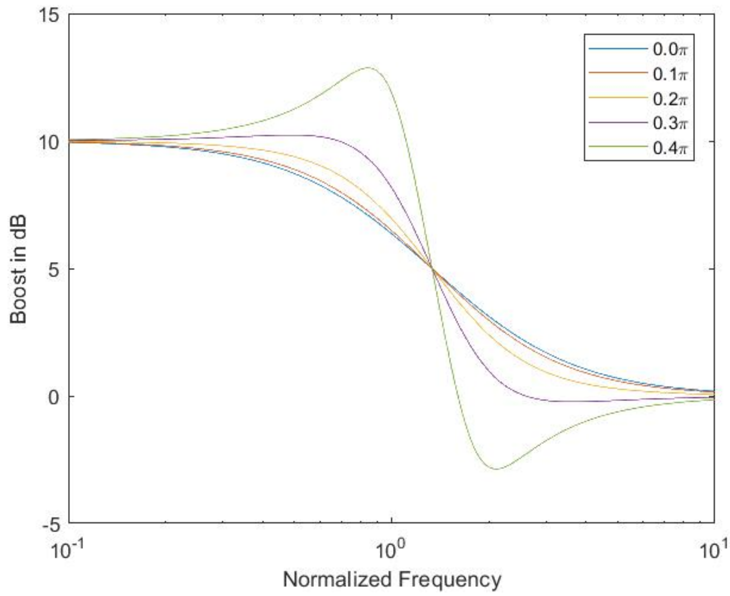

Figure 4: Frequency Response of Bass Boost Filter with Angle Parameter

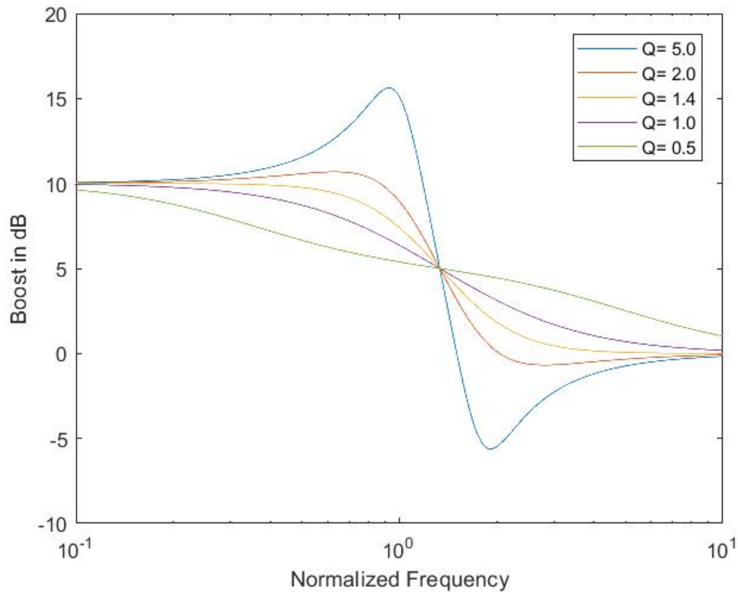

Figure 5: Frequency Response of Bass Boost Filter with Q Factor

where $Q \geq 1$. Peaking occurs for $Q > \sqrt{2}$ and becomes significant for $Q > 2$. Figure 5 shows how the shelf filter frequency response varies with Q. Note that for $Q < 1$, the poles and zeros are on the negative real axis and the filter behaves more like a cascade of two first order filters.

The function `order2filt` derives shelf filters based on Equation 6, adjusting the transition frequency by replacing s with s/ω_0. The shelf filter analog transfer functions are summarized here:

Treble boost

$$H(s) = g\frac{s^2 + 2\omega_0 s/Q + \omega_0^2}{s^2 + 2\sqrt{g}\omega_0 s/Q + g\omega_0^2} \tag{7}$$

Treble cut

$$H(s) = \frac{1}{g}\frac{s^2 + 2\sqrt{g}\omega_0 s/Q + g\omega_0^2}{s^2 + 2\omega_0 s/Q + \omega_0^2} \tag{8}$$

Bass boost

$$H(s) = \frac{s^2 + 2\sqrt{g}\omega_0 s/Q + g\omega_0^2}{s^2 + 2\omega_0 s/Q + \omega_0^2} \tag{9}$$

Bass cut

$$H(s) = \frac{s^2 + 2\omega_0 s/Q + \omega_0^2}{s^2 + 2\sqrt{g}\omega_0 s/Q + g\omega_0^2} \tag{10}$$

where $\omega_0 = 2\pi f_0$, f_0 is the transition frequency, and $g = 10^{|G|/20}$.

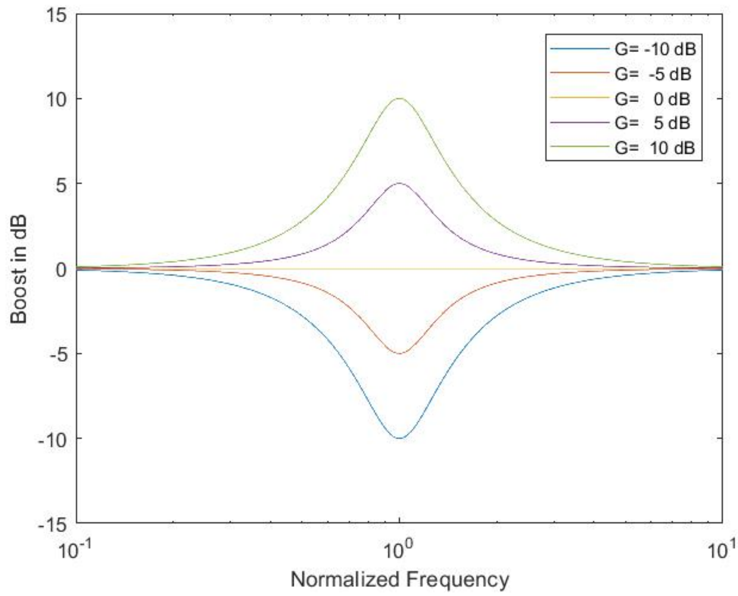

Figure 6: Frequency Response of Peak Filter with $Q = 2$ and Varying Gain

2.2 Peak Filters

Now consider a biquadratic transfer function with poles and zeros given by

$$\begin{aligned} p &= -b \pm j\sqrt{1-b^2} \\ z &= -a \pm j\sqrt{1-a^2} \end{aligned} \tag{11}$$

For $0 \leq a, b \leq 1$, these poles and zeros lie along a circle of radius 1 in the left-hand complex plane. The transfer function is given by

$$H(s) = \frac{s^2 + 2as + 1}{s^2 + 2bs + 1} \tag{12}$$

where the constant factor has been set to $K = 1$. For $a > b$, the corresponding frequency response magnitude, $|H(j\omega)|$, reaches a peak value of a/b at normalized frequency $\omega = 1$. We will let $a = g/2Q$ and $b = 1/2Q$ so that

$$H(s) = \frac{s^2 + gs/Q + 1}{s^2 + s/Q + 1} \tag{13}$$

For $g > 1$, $H(s)$ is a peak filter with a maximum gain of g at normalized frequency $\omega = 1$. Again, if we calculate g from the absolute value of the dB gain G and invert $H(s)$ for $G < 0$, we have a symmetric peak filter that can either boost or cut the signal. Figures 6 and 7 show the effects of both G and Q on the peak filter frequency response.

The analog transfer functions for a peak filter with peak frequency ω_0 become

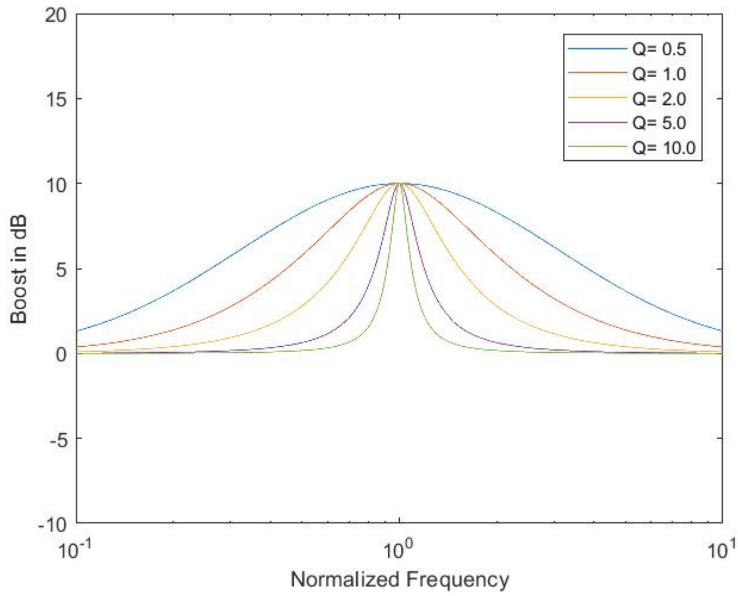

Figure 7: Frequency Response of Peak Filter with $G = 10$ dB and Varying Q

Peak Boost

$$H(s) = \frac{s^2 + g\omega_0 s/Q + \omega_0^2}{s^2 + \omega_0 s/Q + \omega_0^2} \tag{14}$$

Peak Cut

$$H(s) = \frac{s^2 + \omega_0 s/Q + \omega_0^2}{s^2 + g\omega_0 s/Q + \omega_0^2} \tag{15}$$

2.3 Bilinear Transform

As stated earlier, `order2filt` uses the MATLAB toolbox function `bilinear` to convert the analog filter coefficients to digital filter coefficients. The syntax is

```
[b,a]=bilinear(B,A,fs,f0);
```

where `b,a` are the digital filter coefficients, `B,A` are the analog filter coefficients, `fs` is the sampling rate, and `f0` is the peak or transition frequency. Called this way, `bilinear` uses warping to match the frequency responses at the peak or transition frequency. The analog filter coefficients are calculated from the inputs to `order2filter` (see next section) according to the transfer function formulas listed above. We need only use the boost formulas, since we can simply reverse `A` and `B` in the call to `bilinear` for $G < 0$.

3 The `order2filt` Function

3.1 Syntax

```
[b,a] = order2filt(ftype,fs,G,f0,Q);
```

3.2 Description

The function `order2filt` returns the `b` and `a` coefficients of a second order filter. The input `ftype` specifies the filter type, `fs` is the sampling frequency, `G` is the gain in dB, `f0` is the center or transition frequency, and `Q` is the Q factor.

3.3 Inputs

- `ftype` - filter type - `'Bass'` or `'bass'` for bass shelf filter, `'Treble'` or `'treble'` for treble shelf filter, or `'Peak'` or `'peak'` for peak filter
- `fs` - sampling rate in Hz (samples/second)
- `G` - gain in dB. For a peak filter, the gain at the peak frequency, for a bass shelf filter, the gain at frequencies sufficiently below the transition frequency, and for a treble shelf filter, the gain at frequencies sufficiently above the transition frequency
- `f0` - the peak frequency of a peak filter or the transition frequency of a shelf filter
- `Q` - the Q factor - adjusts the slope of a shelf filter and the width of a peak filter

3.4 Outputs

- `[b,a]` - the filter numerator and denominator coefficients - both `b` and `a` are 3-element vectors

3.5 Example

In this example, we use `order2filt` to create a four band equalizer (with all settings fixed) and display the magnitude of the frequency response. The MATLAB script is

```
fs = 44100;
ftype = {'Bass' 'Peak' 'Peak' 'Treble'};
Gain = [-2 1 0 2];
freq = [50 500 2000 5000];
Q = [1.2 1.5 1.5 sqrt(2)];
N = length(Gain);
sos = zeros(N,6);
for n=1:N
    [b,a] = order2filt(char(ftype(n)),fs,Gain(n),freq(n),Q(n));
    sos(n,:) = [b a];
end
[h,f] = freqz(sos,4096,fs);
semilogx(f,20*log10(abs(h)))
xlabel('Frequency in Hz')
```

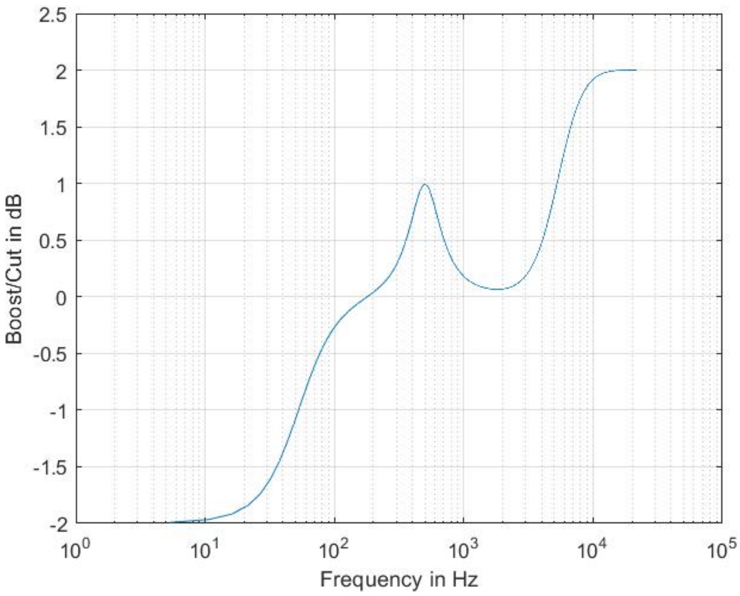

Figure 8: Frequency Response of Four Band Equalizer

```
ylabel('Boost/Cut in dB')
grid on
```

The equalizer is a cascade of four SOS (bass shelf, two peaks, and treble shelf) with dB gains, frequencies, and Q values specified. The four sets of coefficients are calculated using `order2filt` and loaded into an SOS matrix, which organizes the coefficients into four rows of six coefficients. The MATLAB function `freqz` accepts an SOS matrix as a possible input. Figure 8 shows the result.

With the MATLAB DSP system toolbox, you can also use the SOS matrix to creat a dsp.BiquadFilter object, which can filter a stream of audio sample blocks.